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# Unstable horizons

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Abstract: We investigate some of the issues relating to the dynamical instability of general static spacetimes with horizons. Our paper will be partially pedagogical and partially exploratory in nature. After discussing the current understanding, we generalize the proposal of Gubser and Mitra, which identifies dynamical instability of black branes with local thermodynamic instability, to include all product spacetimes with the horizon uniformly smeared over an internal space. We support our conjecture by explicitly exhibiting a threshold unstable mode for Schwarzschild-$AdS_5 \times \mathbf{S^5}$ black hole. Using the AdS/CFT correspondence, this simultaneously yields a prediction for a phase transition in the dual gauge theory. We also discuss implications for spacetimes with cosmological horizons.

Keywords: AdS-CFT and dS-CFT Correspondence, Classical Theories of Gravity, Black Holes.

## Contents





## 1. Introduction

Understanding the stability of a given spacetime is an important issue from many standpoints. In general relativity, stability suggests physical relevance, since a stable static spacetime may be approached dynamically from a reasonably generic set of initial conditions. From a string theory perspective, it is interesting to know which spacetimes are good backgrounds for studying string propagation and which ones have tachyons in the stringy spectrum. This issue gets further bolstered in light of the fact that in certain backgrounds, string propagation is holographically encoded by the dynamics of gauge theories "living at the boundary" of spacetime. In such cases, the instability on the gravitational side is an indicator of interesting gauge theory dynamics, such as phase transitions, etc. Another interesting application relates to closed string tachyon condensation, since an instability in the gravitational background usually signals a physical tachyonic mode in the string spectrum.

While the prospects of answering the question of stability of general spacetimes, in a universal fashion, appear rather dim, it seems much more likely that there could be some underlying principle, which can be used for spacetimes which have horizons. Indeed, encouraged by the successes of such universal laws as those of black hole mechanics, one may be hopeful that black hole spacetimes could be analyzed for (in)stability in a general setting. Therefore, the question we wish to pose is: *What are the necessary and sufficient* *criteria for the onset of a dynamical instability of a horizon?* Apart from deepening our understanding of the properties of horizons, and thereby hopefully shedding more light on quantum gravity, this has further uses in specific contexts. For example, in the context of the AdS/CFT correspondence, one obtains a better understanding of the behaviour of the gauge theory in a typical excited state and its phase transitions. More generally, a universal criterion would not only obviate the need to carry out laborious linear analysis in order to determine stability, but it may be useful even in the cases where such analysis is impossible to carry out because the metric is not explicitly known, but only the appropriate coarse-grained features are understood.

The question of black brane instability was recently addressed by Gubser and Mitra [1], who presented a conjecture for configurations with non-compact translational symmetry along the horizon. Namely, they proposed (and tested numerically) that such a black brane is dynamically unstable to linearized perturbations when it is (locally) thermodynamically unstable. For spacetimes which are holographically dual to some field theory, this is manifested as an instability of the "thermal" ensemble in the field theory. More recently, the proposal received further evidence from Reall [2], who demonstrated the equivalence of the classical lorentzian *threshold unstable mode* — which signals the onset of the Gregory-Laflamme instability — and the euclidean negative mode associated with thermodynamic instability.

While this seems to be a compelling story for the class of configurations considered by Gubser and Mitra, it is desirable to have a criterion which could address more general spacetimes in the absence of translational symmetries or non-compact U(1) symmetries. Not only are the cases with compact U(1)s of interest, but also we would like to consider situations with more complicated symmetries of the extra directions, such as SO(6), a situation pertinent for Schwarzschild Anti-de Sitter black hole which is smeared over a five sphere.

We will see that with a slight modification of the Gubser-Mitra conjecture we can formulate a criterion which will allow us to talk about more general settings. In particular, we will consider all product spacetimes, wherein the black hole is smeared uniformly over some "internal" space. However, we do not yet have a complete picture and our approach will be somewhat experimental in flavour. We will survey the known cases of Gregory-Laflamme like instabilities and attempt to glean from them lessons which are indicative of a pattern. This will lead to our proposal, which we will support using the particular case of Schwarzschild-$AdS_5 \times \mathbf{S}^5$ black hole. We chose this particular case because of its special interest in the context of the AdS/CFT duality. Our main result will lie in establishing the threshold instability for small Schwarzschild AdS black holes. This will simultaneously yield a prediction for the onset of a phase transition in the dual gauge theory.

As further examination of our proposal, we will also briefly consider cosmological horizons. Apart from the need to understand these better within string theory, the intrigue in cosmological horizons stems from the fact that they are observer-dependent. Since even observer-dependent horizons have many properties in common with the genuine spacetime event horizons, we wish to ask whether observer-dependent horizons can also be subject to Gregory-Laflamme instabilities. Of course, this requires the presence of some transverse space. We will consider spacetimes of the form $dS_p \times \mathbf{S}^q$ and $dS_p \times \mathbf{S}^q \times \mathbf{S}^r$. While our conjecture suggests that the former is stable, the surprising result will be that the latter is, in fact, marginally unstable in a special case. While these are neither phenomenologically relevant nor presently realized in string theory, they nevertheless may provide important lessons for future considerations.

The organization of the paper is as follows: In order to motivate our proposal, we will start in section 2 by reviewing the known examples of Gregory-Laflamme instabilities. We will then formulate a slightly modified version of the Gubser-Mitra conjecture in section 3, and test it in section 4 by studying the example of AdS Schwarzschild black holes. Having established threshold instability for such systems, we will briefly comment on its implications for the dual gauge theoretic description. We will then consider our conjecture in the context of cosmological horizons in section 5, and conclude in section 6 with a summary and some speculations about more general spacetimes.

## 2. Examples of unstable horizons

Ideally, given a general spacetime with a horizon, one would like to find and understand a universal criterion for the horizon to be unstable to perturbations.[1] In the absence of some immediate general principle which would guide us, we may consider the hints that already appear in the literature. We have several "data points", i.e., statements about specific black holes or classes of black holes, which may provide clues to the generic case with arbitrary event horizon. These usually involve a one-parameter family of solutions to Einstein's equation and specify the value of the parameter which separates the classically stable solutions from the unstable ones. Rather than adhering to the chronological development of ideas, we will discuss these in order of increasing conceptual difficulty.

### 2.1 Noncompact, translationally invariant cases

Perhaps the simplest class of horizons to consider are those exhibiting noncompact translational invariance, namely black branes. Recently, a very concrete proposal has been suggested by Gubser and Mitra [1], for this subclass of event horizons. The Gubser-Mitra conjecture states that for black branes with noncompact translational symmetry, the Gregory-Laflamme instability, i.e., a tachyonic mode in small perturbations, is equivalent to a local[2] thermodynamic instability, as manifested by the Hessian matrix having positive eigenvalues.[3] To test their conjecture, [1] study particular plane-symmetric $AdS_4$-

[1]Of course, this assumes an affirmative answer to the question "is there a universal criterion?" As mentioned earlier, the main reason for the authors' optimism stems from the lesson that horizons exhibit unexpectedly universal behaviour.

[2]Gubser and Mitra use the terminology "local", to distinguish it from the global thermodynamic instability, given by the existence of a more entropically favorable configuration.

[3]Let us elaborate a little on this scenario. As [1], we will work in the microcanonical ensemble, with fixed energy or energy density. Then it is most natural thermodynamically to consider the second variation of the entropy with respect to energy or mass to be related to specific heat, a concept familiar in the canonical ensemble. In general, one could also consider chemical potentials and fix other charges; then the information about the stability of the ensemble is encoded in the corresponding Hessian matrix. Specifically, for conserved quantities (including the energy) $\{Q_i\}$, the Hessian matrix is given by $H_{ij} \equiv \frac{\partial^2 S}{\partial Q_i \partial Q_j}$.

Reissner-Nordström black branes. In the system considered, the instability is manifested in the linearized approximation, not by a growing mode of metric perturbations, but rather by a growing mode of the vector potential perturbations, which are easier to handle computationally. This gives evidence for the conjecture to a high numerical accuracy.

Further evidence for the conjecture was provided by Reall [2], whose argument relies on the fact that we can think of a translationally invariant black brane solution as a black hole which is smeared homogeneously along the extra directions. Furthermore, some black hole geometries, when thought of in euclidean signature, have a negative eigenmode for the euclidean Lichnerowicz operator. This mode typically has no relevance for the dynamics of the black hole spacetime, because the lorentzian equations are different from the euclidean Lichnerowicz operator eigenvalue equations. However, if we consider only static lorentzian perturbations about the black string background, then we can reduce the lorentzian problem to the euclidean eigenvalue equation. This is what Reall proposes to call the *threshold unstable mode*, since it is the mode which, in the lorentzian geometry, signals the onset of the instability.

To give an explicit example, consider the case of a black string in five-dimensions. This is just the four-dimensional Schwarzschild black hole whose horizon is smeared along a non-compact translationally invariant fifth direction. The corresponding euclidean continuation of the four-dimensional black hole has a mode with negative eigenvalue. This is precisely the mode discovered in the context of determining the stability of hot flat space, by Gross, Perry, and Yaffe [3]. The eigenvalue equation for the euclidean metric fluctuations is

$$\Delta_E h_{\mu\nu} = \lambda_E \, h_{\mu\nu} \tag{2.1}$$

with $\lambda_E < 0$. For time independent fluctuations in lorentzian signature, the equations we obtain are just the same as the euclidean ones, since $\Delta_L(static) = \Delta_E$. However, one does not associate any meaning to the negative eigenmode in the lorentzian context, because the fluctuation equation corresponds to the zero eigenvalue case: $\Delta_L h_{\mu\nu} = 0$. So the euclidean negative mode has no role to play in the dynamical evolution of the black hole spacetime.

However, if we now have some extra directions transverse to the black hole, as in the black string case, then we could consider fluctuations which carry momentum along the transverse directions. Namely, $h_{\mu\nu}(x,y) = h_{\mu\nu}(x)e^{iky}$, where $x$ denotes the coordinate in the black hole directions and $y$ denotes the transverse $\mathbb{R}$ coordinate. Then the fluctuations are governed by the equation

$$\Delta_L h_{\mu\nu}(x) + k^2 \, h_{\mu\nu}(x) = 0 \,. \tag{2.2}$$

This is now identical to the euclidean eigenvalue equation (2.1), with the role of the eigenvalue being played by the momentum carried by the fluctuations in the transverse directions, $\lambda_E \equiv -k^2$. Clearly, there exists a critical momentum, which coincides precisely with the eigenvalue one calculates in the euclidean spacetime. This gives the threshold unstable mode.

The relation between the euclidean negative mode of the black hole and the lorentzian static mode of the black string was actually noticed much earlier by Gregory and Laflamme [4]. Reall [2] elaborates on this, and, working in the canonical ensemble, shows

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that a black hole with negative specific heat cannot be stable and therefore must have a euclidean negative mode. Since the local thermodynamic instability is manifested by negativity of specific heat in the canonical ensemble, and equivalently by the positivity of the Hessian matrix in the microcanonical ensemble, this explains the connection between classical (lorentzian) stability and thermodynamic stability of black branes conjectured by Gubser and Mitra [1]. Later, Gregory and Ross [5], who considered neutral black branes in finite cavity, provided a more complete argument that the euclidean negative mode (and therefore the classical instability) occurs precisely when the black brane is thermodynamically unstable. However, no complete proof of the universal equivalence between the euclidean negative mode and local thermodynamic instability has been established so far.

While the Gubser-Mitra proposal yields a rather nice picture relating the thermodynamic and the classical dynamical instabilities, conceptually it leaves something to be desired. The former, manifested e.g., by the specific heat being negative, pertains to the dimensionally-reduced black hole, whereas the Gregory-Laflamme instability manifests itself by breaking translational invariance in the extra dimensions, which one would associate rather with the properties of the transverse space. With [1] 's restriction to noncompact translationally invariant cases, the only "property" of the transverse space that one is free to vary is the dimensionality.

Given the success of the conjecture for the noncompact translationally invariant branes, one may be tempted to try to generalize the conjecture further by postulating a more universal applicability. However, such naive approach fails at once, because of several known counter examples.

### 2.2 Compactified black string

The earliest and best-known example of the Gregory-Laflamme instability is the black string with a compact horizon, i.e., the $\mathrm{Schw}_4 \times \mathbf{S^1}$ geometry. Specifically, in static coordinates, the metric is

$$ds^2 = -\left(1 - \frac{r_+}{r}\right) dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} dr^2 + r^2\, d\Omega_2^2 + dz^2\,, \tag{2.3}$$

where $z$ is identified with $z + L$, i.e. the length of the compact direction is $L$, and the black hole horizon radius is $r_+$. For the noncompact $L \to \infty$ case, [6] found that in perturbing this geometry, all modes with sufficiently long wavelength (of the order of the 4-d black hole radius, specifically $\lambda \geq 7.1\, r_+$) are dynamically unstable. In the compactified case, the analysis is identical, except that now the effective masses of the higher modes will be quantized in units of $2\pi n/L$. Equivalent picture, but one perhaps more useful conceptually, is to fix the length of the compact direction $L$ and vary $r_+$. Then the Gregory-Laflamme analysis shows that only sufficiently small black holes will become dynamically unstable to clumping in the compact direction.

At the first glance, this may seem inconsistent with the Gubser-Mitra conjecture, since the 4-dimensional Schwarzschild black hole has positive Hessian matrix $\frac{\partial^2 S}{\partial E^2}|_{r_+} = 8\pi > 0$, or equivalently, negative specific heat for all values of its radius, $C_v \equiv \frac{\partial E}{\partial T}|_{r_+} = -2\pi L r_+^2 < 0$, so that one would naively expect that all black holes should exhibit a corresponding dynamical instability. In fact, there is no contradiction, simply because the conjecture does not apply in this case: while translational invariance is satisfied, the extra direction is compact. In the noncompact limit, corresponding to the length of the $S^1$ getting large, $L \to \infty$, the Gubser-Mitra proposal is satisfied, since all black holes, having finite radius, are "small" (i.e., $r_+ \ll L$) and therefore unstable.

This means that while there is a euclidean negative mode for all values of the black hole radii $r_+$ (for a fixed $L$), only for sufficiently small $r_+$ is this negative mode associated with a dynamical instability in lorentzian time. It is a very interesting question, how this euclidean negative mode is manifested in the absence of any dynamical instability.

Gregory and Laflamme also noted that the entropy of a fully 5-dimensional black hole[4] is greater than that of the black string with same mass, when $L$ is large enough. This suggested that an instability of the kind found by their linearized perturbation analysis may initiate the fragmentation of the horizon, so that the black string pinches off and becomes a 5-dimensional black hole. Such a process would be entropically favorable in roughly the same regime as where the 5-d black hole could "fit" in the compact direction, namely for small black holes.

This picture of black string fragmentation was widely accepted until recently, when Horowitz and Maeda [7] reexamined the fate of the instability, with very surprising and intriguing outcome: Instead of fragmenting into disconnected pieces, the horizon can only deform. It cannot pinch off, because any $\mathbf{S}^2$ on the horizon cannot shrink to zero size.[5] Given this, the conjecture for the black string is that it settles down to a new static black string solution which is not translationally invariant along the horizon. Nevertheless, (the converse of) the entropy argument may still be used as some indicator of whether an instability is possible: when the entropy of the 5-d black hole is smaller than that of the corresponding black string configuration, we would not expect any deformation of the black string to be entropically favorable, so that the black string should be stable.

For completeness, we present the specific calculation of the transition point, i.e., the relation between $r_+$ and $L$, for which the black string and the 5-d black hole of the same mass also have the same entropy. Let us approximate[6] the 5-d black hole by the ordinary $\mathrm{Schw}_5$ metric, with 5-d radial coordinate $R$ and the horizon radius $R_+$:

$$ds^2 = -\left(1 - \frac{R_+^2}{R^2}\right) dt^2 + \left(1 - \frac{R_+^2}{R^2}\right)^{-1} dR^2 + R^2\, d\Omega_3^2\,. \tag{2.4}$$

The masses of the black string and this 5-d black hole are, respectively, $M_{bs} = \frac{1}{2} r_+ L$ and $M_{bh} = \frac{3\pi}{8} R_+^2$, while the respective horizon areas are: $A_{bs} = 4\pi r_+^2 L$ and $A_{bh} = 2\pi^2 R_+^3$.

[4] I.e., where a spacial slice of the horizon has topology of $\mathbf{S}^3$, rather than $\mathbf{S}^2 \times \mathbf{R}$, as for the black string.

[5] This was proved by [7] only for finite proper time using the Raychaudhuri equation, but strong arguments were presented even for the infinite time case. Their argument in fact holds for general horizons, which we will make use of later.

[6] This will not be the exact solution because of the finite size of the compact direction. Physically, one may think of this in the decompactified picture as the horizon being deformed by an infinite periodic array of black holes. However, it will turn out that the separation of the 5-d black holes at the critical point is large enough to lend us some confidence that such approximation is still reasonable.

Equating the masses and the horizon areas then yields the transition point,

$$r_+ = \frac{2}{3}R_+ = \frac{16}{27\pi}L\,. \tag{2.5}$$

Note that $L \approx 3.53R_+$, so that the 5-d black hole can easily fit in the compact direction.[7]

Let us now compare this value with the actual onset of instability as determined by the linearized analysis of [6]. If we fix $L$ and vary $r_+$, we find that the black string becomes entropically unfavorable for $r_+ < 0.189L$, but it becomes actually unstable only for $r_+ < 0.14L$. This shows the important point that while the global entropy argument can reveal where an instability is allowed to occur, it does not tell us where it actually does occur. In other words, the existence of a more entropically favorable configuration does not require an instability of the initial configuration.

An alternate formulation of the problem which is more analogous to the one we will employ later, is to consider both the parameters $r_+$ and $L$ fixed, and look for unstable modes as perturbations of this spacetime. Then the criterion for finding unstable modes is a bit different, since these have to be consistent with the spacetime, i.e., the wavelength has to be quantized: $\lambda = L/n$ for some integer $n$. Suppose now that we are in the stable regime with $r_+ > 0.14L$. Despite the existence of a negative euclidean mode (i.e., thermodynamic instability), there is no lorentzian mode which would be unstable. Why? Simply because we would need the wavelength of such a mode to be sufficiently large, $\lambda > 7.1r_+$, but this is incompatible with $\lambda = \frac{L}{n} < \frac{7.1}{n}r_+$.

This reiterates the point mentioned earlier, which is obvious for ordinary Schwarzschild black holes, that thermodynamic instability does *not* generally imply a dynamical instability. At first glance, there is no reason to expect them to be related at all, since the former has quantum roots while the latter concerns a purely classical evolution; it may even seem rather surprising that there in fact is a deep relation for the noncompact cases.

### 2.3 Black string in a box

The above example has demonstrated that if the dimensionally reduced black hole has negative specific heat, the corresponding black string may or may not be unstable, depending on the size of the compact direction. Let us now consider the converse situation, namely what happens when the dimensionally reduced black hole has positive specific heat. One way to achieve this is by considering charged black holes as in [1]; but a simpler way is to put the black hole in a finite cavity, as done in e.g. [5].

For noncompact translationally invariant solutions, [5] find that an uncharged black brane in a spherical cavity is classically unstable if and only if it is locally thermodynamically unstable, which provides further supporting evidence of the Gubser-Mitra proposal.

[7] As an aside, this relation becomes tighter as we increase the extra dimension; for example, carrying out the above calculation for transition point between 5-d black hole $\times\mathbb{R}^1$ and approximate 6-d black hole yields $r_+ = \frac{3}{4}R_+ = \frac{3^5}{2^{10}}L$, so that $L \approx 3.16R_+$, i.e., $2R_+/L \approx 0.63 < 1$. For high dimensions, this ratio eventually gets "dangerously" big: for $d = 10$, $2R_+/L \approx 0.91$, and for $d = 26$, $2R_+/L \approx 1.48 > 1$, so the critical black hole would no longer fit into the extra direction, and our approximation breaks down. However, the actual instability, as calculated numerically by Gregory-Laflamme, occurs for smaller black holes than the critical ones, and these do seem to fit.

The important point to note here is that the presence of a boundary affects the classical instability. While this may seem rather odd if one thinks of the Gregory-Laflamme instability as pertaining to the horizon, the reason for this behaviour is that the boundary conditions restrict the allowed initial perturbations, just as the geometry of the internal space in the previous subsection.

Considering black holes or black branes in a box has many similar features to considering these black objects in appropriate asymptotically Anti de Sitter spacetimes — after all, the AdS spacetime has a confining potential which acts like a box of size given by the AdS radius. Therefore, we may expect many similar features for these cases. In particular, as discussed below, the specific heat changes sign as the ratio of the black hole radius to the AdS radius varies. We will see in section 4 to what extent these expectations are realized.

## 3. Conjecture for product spacetimes

What we have just seen in the survey of various examples gives us a little flavour of what to expect in general. First of all, it is clear that a simple criterion based on global entropy arguments does not suffice to determine a Gregory-Laflamme like instability, as manifested clearly by the black string example. At best, it indicates the presence of a more entropically favourable configuration, which may be reachable only by quantum tunneling. To wit, there could be instanton solutions, but the classical fluctuations need not be tachyonic.

On the other hand, a completely local thermodynamic criterion, such as that proposed by Gubser and Mitra, is also unlikely to hold in more general settings than those considered by the authors, namely when the transverse space is not $\mathbb{R}^n$. The simple reason is that such a criterion applies only to the dimensionally reduced black hole, whereas the Gregory-Laflamme instability explicitly involves the transverse space over which the black hole is initially smeared. Hence, if we wish to consider arbitrary transverse spaces, the proposed criterion should include the information about them somehow. In fact, from our preceding discussion, we can guess such a criterion rather easily.

In particular, any viable proposal seems necessarily to involve some features that are local (in the thermodynamic sense), but at the same time, also needs to have some global information, in the sense of knowing about the full spacetime. Taking the Gubser-Mitra criterion as the right local criterion, one is then led to propose the following:

Consider a static solution to Einstein's equations (with the stress tensor satisfying all the necessary requirements), which is of the form $\mathbf{B}_m \times X^n$, where $\mathbf{B}_m$ is an $m$-dimensional black hole spacetime and $X^n$ is an $n-$dimensional "transverse" space; namely, the horizon has a direct product structure with spatial slice $\mathbf{S}^{m-2} \times X^n$. If $\mathbf{B}_m$ has a euclidean Lichnerowicz operator with a negative eigenmode, i.e.

$$\Delta_E(\mathbf{B}) h_{\mu\nu} = \lambda_E \, h_{\mu\nu}$$

with $\lambda_E < 0$, and if the usual Laplacian on $X^n$ has eigenfunctions $\phi$ with eigenvalues $-\mu^2$ (which can be discrete), i.e.,

$$\nabla^2(X)\phi = -\mu^2 \, \phi \,,$$

then we expect the onset of instability to occur exactly when we can saturate the euclidean eigenmode with the 'momentum' in the $X^n$ directions. Namely, *the threshold unstable mode will occur when* $\mu^2 = -\lambda_E$. Furthermore, we expect that for lower momenta, $0 < \mu^2 < -\lambda_E$, there will be dynamical instability manifested by modes growing exponentially in time.

Stated differently, for black hole spacetimes having a direct product structure, not only is it required (in the language of Gubser and Mitra) that there be an instability of the thermal ensemble as manifested by positive eigenvalues of the Hessian matrix, but also that the restrictions the boundary conditions at 'infinity' place on the set of allowed eigenfunctions in the transverse directions (those given by $X^q$), be satisfied.

Several comments about our conjecture are in order. First, although the criterion we propose is a very simple modification of the Gubser-Mitra conjecture [1], it applies to a much larger class of solutions. In our notation, [1] required the transverse space $X^n$ to be $\mathbb{R}^n$, whereas for us it can be anything (subject to the field equations). Second, our proposal holds for all the hitherto considered examples, including the compact black string as well as the black string in a box. Clearly, the Gubser-Mitra conjecture is a special case (since one can always choose the mode with arbitrary $\mu^2$ on $\mathbb{R}^n$, and therefore compensate for any $\lambda_E$), so that all these cases are satisfied as well. From an aesthetic point of view, it is nice to note that a sufficient admixture of local and global conditions seem to be necessary to cover a larger set of examples. Our proposal is formulated in a fully geometrical fashion and takes into account the full spacetime, while at the same time it does not rely on the global entropy arguments.

As mentioned above, our criterion would greatly facilitate the stability analysis of black hole product spacetimes, since one would no longer need to perform the explicit linear analysis.[8] Also, note that apart from dictating where one should expect an instability, an immediate consequence of our proposal is that any black object, which can be written in a product fashion as above, with positive eigenvalues of the euclidean Lichnerowicz operator, has to be dynamically stable.

The proposal is most justifiable for the threshold unstable mode. More interesting, but also more uncertain, aspect relates to the dynamical instability. Clearly, lacking the simple relation between the euclidean and the lorentzian picture for dynamical settings, we don't have any clear proof of the presence of a growing unstable mode for any $0 < \mu^2 < -\lambda_E$ (although it seems to hold in the known cases). Finally, while a step forward, our conjecture does not address the most general spacetime with horizon. In particular, there are many interesting examples which cannot be written in terms of a direct product. At present, we do not have any proposal pertaining to these cases, though we now briefly mention one special class of generalizations.

We have in the above restricted our attention to black holes smeared over an internal space homogenously. One could also consider examples wherein the black holes are warped

[8] Although establishing local thermodynamic instability would be easier still compared to computing the euclidean negative mode, we refrain from formulating our conjecture in that language, because of the prevailing difficulty in associating the euclidean negative mode to the local thermodynamic instability in these more general cases.



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over an extra dimension.[9] In these examples, too, the static mode can be decomposed as a part coming from the warping direction and a part in the black hole spacetime, i.e., we simply choose a factorised ansatz for our perturbations. Then the fluctuation equations split up, as in the above case, to an eigenvalue problem. However, now the analog of the 'momentum' from the internal directions which plays the role of the euclidean eigenvalue is determined not in terms of simple eigenvalues of the Laplacian on the internal space, but in terms of the energy levels of a quantum mechanics problem, where the potential is a function of the warp fator. Some examples along these lines have been discussed in [8, 9]. Generalizations to the cases where we have both warping and homogeneous smearing over an internal space should be obvious superpositions of the individual cases.

## 4. Schwarzschild-$AdS_5 \times \mathbf{S^5}$

Let us now apply our conjecture to a spacetime which has been outside the scope of the Gubser-Mitra conjecture, namely the Schwarzschild black hole in $\mathrm{AdS}_5$ times a five sphere, which is a solution to the type-IIB equations of motion. In particular, unlike the previous cases with translational symmetry discussed in section 2, this is an example where the solution is neither translationally invariant nor is the symmetry group along the transverse directions noncompact. Rather, since the extra directions are along the 5-sphere, the original solution has SO(6) symmetry. The metric is

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3^2 + L^2 d\Omega_5^2 \tag{4.1}$$

where, for Schwarzschild-$\mathrm{AdS}_5$, $V(r) = 1 + r^2/L^2 - \frac{r_+^2}{r^2}(1 + r_+^2/L^2)$; $r_+$ is the horizon radius and $L$ is the radius of both AdS and the 5-sphere.

We will now motivate our present guess of where we would expect the instability to appear. The first observation is that the Schw-$\mathrm{AdS}_5$ black hole is thermodynamically unstable for $r_+ < L/\sqrt{2}$. This can be seen either by considering the Hessian matrix,[10] or more simply by evaluating the specific heat: The Hawking temperature is

$$T = \frac{2r_+^2 + L^2}{2\pi r_+ L^2} \tag{4.2}$$

so $C_v \sim dT/dr_+ = \frac{1}{2\pi L^2 r_+^2}(2r_+^2 - L^2) < 0$ if $r_+ < L/\sqrt{2}$, independent of the details of the compact part of the geometry. This in fact matches precisely with the Schwarzschild-AdS black hole euclidean mode becoming negative, as conjectured by [10] and demonstrated explicitly by Prestidge [11].

However, this does not provide the true criterion for the dynamical instability, as explained in the previous section. All that the local thermodynamic stability calculation shows is that a black hole with $r_+ > L/\sqrt{2}$ should be dynamically stable. The point is

[9] We would like to thank G. Kang for bringing this to our attention.

[10] In particular, $M \sim S^{2/3}(S^{2/3} + L^2(\pi^2/2)^{2/3})$, so that the solution is thermodynamically unstable if $\frac{\partial^2 M}{\partial S^2} < 0$, i.e., $S^{2/3} < \frac{1}{2}(\pi^2/2)^{2/3}L^2$. Since $S = \frac{2\pi^2}{4}r_+^3$, this translates into $r_+ < L/\sqrt{2}$.



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that the unstable mode simultaneously has to be an eigenmode of the Laplacian on the 5-sphere. In particular, we expect the lowest allowed $\ell$ (denoting the harmonic on the $\mathbf{S}^5$) for which unstable mode can exist to determine the parameters for the onset of the dynamical instability.

### 4.1 Threshold unstable mode

To wit, in order to establish the existence of a static mode that signals the onset of the instability, we shall follow the treatment of [12, 13] to obtain the fluctuation equation about the background solution. Our starting point is the background corresponding to the smeared Schwarzschild AdS black hole with the metric given by (4.1). In addition, one also has a non-trivial 5-form background. We shall consider only the fluctuations of the metric on the AdS part of the spacetime. The fluctuation of the metric components along the five sphere directions are scalars from the $\mathrm{AdS}_5$ point of view, so these should have no exponentially growing fluctuations [14]. Since the traceless tensor fluctuations on $\mathrm{AdS}_5$, corresponding to the graviton mode, decouple from the 4-form fluctuations (cf. [12, table II]), our ansatz also assumes that at leading order, the fluctuations of the RR 4-form potential can be consistently set to zero. This is analogous to the black string example of Gregory and Laflamme [6], where the corresponding equations have the same structure.

The fluctuation equations can be obtained by expanding the equations of motion of type-IIB supergravity about this background. Since we are looking for metric fluctuations of the AdS part, we can restrict ourselves to looking at Einstein's equations with indices along the AdS directions. Adopting the notation of [13], we write the full linearized Ricci tensor as

$$\begin{aligned} R^{(1)}_{MN} = -\frac{1}{2}\Big[ &\left(\nabla_x^2 + \nabla_y^2\right) h_{MN} + \nabla_M \nabla_N h^P{}_P - \nabla_M \nabla^P h_{PN} - \nabla_N \nabla^P h_{PM} - \\ &- 2R_{MPQN} h^{PQ} - R^P{}_M h_{NP} - R^P{}_N h_{MP} \Big] , \end{aligned} \tag{4.3}$$

where $\nabla_x^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $\nabla_y^2 \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ are the d'Alambertian operators on $\mathrm{AdS}_5$ and $S^5$, respectively. Since the equation of motion is

$$R^{(1)}_{MN} = \frac{1}{48} F_{MP_2P_3P_4P_5} F_N^{P_2P_3P_4P_5} - \frac{1}{480} g_{MN} F^2_{(5)} , \tag{4.4}$$

the fluctuation equation for the AdS part simplifies to

$$-\frac{1}{2}\Big[ \left(\nabla_x^2 + \nabla_y^2\right) h_{\mu\nu} + \nabla_\mu\nabla_\nu h^\rho{}_\rho - \nabla_\mu \nabla^\rho h_{\rho\nu} - \nabla_\nu \nabla^\rho h_{\rho\mu} - 2R_{\mu\rho\sigma\nu} h^{\rho\sigma} R^\rho{}_\nu h_{\rho\mu} \Big] + \frac{4}{L^2} h_{\mu\nu} = 0 . \tag{4.5}$$

We make the following choice as our ansatz for metric with the fluctuations:

$$\begin{aligned} ds^2 = &-V(r)\left(1 + \epsilon\psi(r) Y_\ell(\Omega_5)\right) dt^2 + \frac{1}{V(r)}\left(1 + \epsilon\chi(r) Y_\ell(\Omega_5)\right) dr^2 + \\ &+ r^2\left(1 + \epsilon\kappa(r) Y_\ell(\Omega_5)\right) d\Omega_3^2 + L^2\, d\Omega_5^2 ; \end{aligned} \tag{4.6}$$

without loss of generality, we can set $L \equiv 1$ and subsequently measure all lengths in terms of the AdS units. Here $Y_\ell(\Omega_5)$ denote the usual spherical harmonics on the five sphere, satisfying $\nabla_y^2 Y_\ell(\Omega_5) = -\ell(\ell+4)\, Y_\ell(\Omega_5)$. In addition we choose a transverse traceless gauge for the fluctuations,

$$\nabla^\mu h_{\mu\nu} = 0\,, \qquad h^\mu{}_\mu = 0\,,$$

which implies

$$\begin{aligned}
\kappa(r) &= -\frac{1}{3}\left(\psi(r) + \chi(r)\right) \\
\psi(r) &= \frac{2rV}{rV' - 2V}\,\chi'(r) + \frac{rV' + 8V}{rV' - 2V}\,\chi(r)\,.
\end{aligned} \tag{4.7}$$

The equation of motion for the $(rr)$ component is:

$$\begin{aligned}
-V\chi''(r) &+ \left\{\frac{2r^2(VV'' - V'^2) - 3rVV' + 10V^2}{r(rV' - 2V)}\right\}\chi'(r) + \\
&+ \left\{\frac{r^2V'V'' + r(8VV'' - 7V'^2) + 4VV'}{r(rV' - 2V)}\right\}\chi(r) = -\ell(\ell+4)\,\chi(r)
\end{aligned} \tag{4.8}$$

where we have eliminated the functions $\psi(r)$ and $\kappa(r)$ using the gauge conditions (4.7). In principle we have another equation from the $(tt)$ component, which we present here for completeness:

$$\begin{aligned}
V\psi''(r) + \left(V'(r) + \frac{3V}{r}\right)\psi'(r) - \left(\frac{V'^2}{2V} - \frac{V'}{r}\right)\psi(r) - \left(V'' - \frac{V'^2}{2V} - \frac{V'}{r}\right)\chi(r) &= \\
= -\ell(\ell+4)\,\psi(r)
\end{aligned} \tag{4.9}$$

When we use the second gauge condition, this becomes a third order differential equation for $\chi(r)$; however, the solutions of this equation are different from those of (4.8) only by a pure gauge term. Similarly, all the components of (4.5) are either trivial or physically equivalent to (4.8).

Hence, to find the metric fluctuations, we can solve eq. (4.8), subject to the physical boundary conditions: regularity everywhere (including the horizon) and normalizability given by appropriate fall-off at infinity. Having solved for $\chi(r)$, we can then use (4.7) to obtain the other components of the metric fluctuation. This defines an eigenvalue problem: for every $r_+$, we can find the value of the real (but not necessarily integral) number $\ell$, for which there exists a solution $\chi(r)$ satisfying the boundary conditions. Alternately, for any fixed $\ell$, we can find the corresponding value of $r_+$, which is physically more relevant for the problem at hand. Thus, rather than finding the explicit form of the metric fluctuation, we wish to find the 'eigenvalue' $r_+$, for $\ell = 1, 2, 3, \ldots$

However, before presenting the results, let us first make a few comments about the $\ell = 0$ case. Using the ansatz $L = 1$, $\ell = 0$, $r_+ = 1/\sqrt{2}$, we can actually find an analytic solution to (4.8). The metric fluctuations (as given by (4.6)) are:

$$\psi(r) = \frac{2r^2 + 1}{2r^4(2r^2 + 3)}\,, \qquad \chi(r) = \frac{1}{r^4(2r^2 + 3)}\,, \qquad k(r) = -\frac{1}{6r^4}\,. \tag{4.10}$$

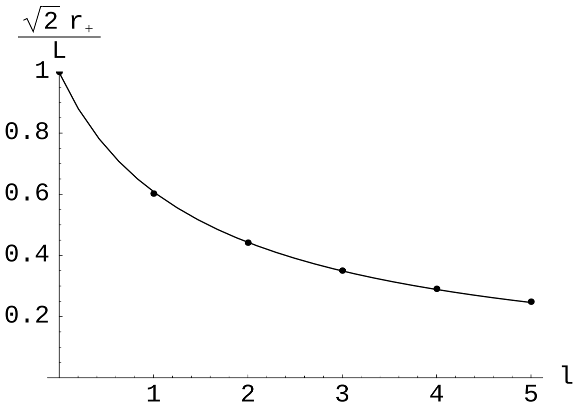


**Figure 1:** For Schw-$AdS_5 \times \mathbf{S}^5$ black holes, $\sqrt{2}/Lr_+$ as a function of $\ell$, with the corresponding fit to guide the eye.

One can easily verify that this satisfies all the regularity constraints and falls off sufficiently fast at infinity. We may then ask, why is this not the first threshold unstable mode? The answer is that $\ell = 0$ corresponds to the zero mode on the 5-sphere, so such deformation would not represent a Gregory-Laflamme instability. Also, we know that in the Kaluza-Klein reduced picture, AdS black holes are stable. One may still worry that, in this 5-dimensional picture, our (linearized) solution would violate the black hole uniqueness; however, one can easily check that with suitable change of coordinates, it corresponds to the standard Schwarzschild-$AdS_5$ black hole with slightly modified mass. Hence, we learn that this was a rather special case: Only at this point did the first variation in temperature with respect to $r_+$ vanish, while for all other values of $r_+$ (or equivalently $\ell$) the mass remains fixed even at linear order.

We now present the results for the higher values of $\ell$, which correspond to genuine threshold unstable modes. Table 1 lists the values of $r_+$ for $\ell = 1, \ldots, 5$, and we include the corresponding euclidean negative mode, $\lambda_E = -\ell(\ell+4)$.

| $\ell$ | $\lambda_E$ | $r_+$ |
|---|---|---|
| 1 | $-5$ | 0.4259 |
| 2 | $-12$ | 0.3214 |
| 3 | $-21$ | 0.2478 |
| 4 | $-32$ | 0.2056 |
| 5 | $-45$ | 0.1759 |

**Table 1:** Values of the black hole radius $r_+$ corresponding to the threshold unstable mode for the first few harmonics $\ell$.

We also plot $\frac{\sqrt{2}}{L}r_+$ as a function of $\ell$ in figure 1, where we included a fit to guide the eye better. This rescaling makes it easy to see that as $\ell \to 0$, $r_+ \to L/\sqrt{2}$, which corresponds to the onset of the local thermodynamic instability.

Not surprisingly, (4.8) is the same equation as that obtained by Prestidge [11], who examined the eigenvalues of the euclidean Lichnerowicz operator. In his study, the rhs of (4.8) had $-\ell(\ell+4)$ replaced by $\lambda_E$, the euclidean eigenvalue. Prestidge found that for $r_+ > L/\sqrt{2}$ the eigenvalue was positive, at $r_+ = L/\sqrt{2}$ there was a mode with zero euclidean eigenvalue, and for smaller black holes there was a negative eigenvalue. The eigenvalue $\lambda_E$ appeared to be a monotonically increasing function of $\rho_+ \equiv \sqrt{2}r_+/L$, going to negative infinity as $\rho_+ \to 0$. Furthermore, [11] also plotted the euclidean eigenvalue $\lambda_E$ as a function of $\rho_+$. All this agrees with our results, as one can see from figure 1.

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Hence, we have established the existence of the static mode in the lorentzian geometry which marks the separation between the stable and the unstable regime. We have verified numerically that the above equation, defining a well-posed eigenvalue problem, has a solution for various values of $\ell$.

### 4.2 Gauge theory implications

Part of the motivation for considering the Schwarzschild-$AdS_5 \times \mathbf{S}^5$ solution of course rests on the AdS/CFT correspondence [15]–[18]. Let us now consider what all this implies for the corresponding dual 4-dimensional SYM gauge theory.

Large black holes in AdS are described by an approximately thermal state, with temperature T given by (4.2), in the gauge theory. However, small black holes (compared to the AdS radius) do not have any role in the thermal ensemble. The reason is the following: When we are considering thermal Yang-Mills theory on $\mathbf{S}^3$, we are instructed to look for five dimensional spacetimes whose boundaries are $\mathbf{S}^3 \times \mathbf{S}^1$. There are two spacetimes which fit this criterion [10, 19]; one is the Schwarzschild black hole in AdS and the other is a hot thermal gas in AdS. Which of these geometries contributes dominantly to the field theory is decided in terms of their contribution to the free energy as per the euclidean path integral. The black hole contributes dominantly for high temperatures, while the low temperature behaviour is governed by the thermal gas in AdS space. The transition occurs at $r_+ \sim L$; in the field theory side this is best thought of as a deconfinement transition in large N gauge theory [19]. Since the thermal gas starts to dominate the saddle point of the euclidean path integral at about the point where the black holes in AdS start having negative specific heat, it is unclear what the role of the small black holes is in the thermal ensemble. This transition is the so-called Hawking-Page transition [10], and is quite distinct from the Gregory-Laflamme instability which we are interested in. As in the previous examples, we shall work in the microcanonical ensemble, where we can ask the question about the stability of the small black holes in AdS.

Let us first consider the small black holes with $L/\sqrt{2} > r_+ > 0.426L$. While these are locally thermodynamically unstable, as argued above, they are still dynamically stable. Although the corresponding state in the gauge theory is no longer the thermal state, it still maintains the full SO(6) symmetry. Now we come to the black hole with $r_+ = 0.426L$. For such a black hole we demonstrated that the graviton fluctuation, which is a dipole on the $\mathbf{S}^5$, leads to a static threshold unstable mode. This means that the dynamical instability wants to deform the homogeneously smeared horizon along the $\mathbf{S}^5$ into an inhomogeneous solution, with the horizon shape being deformed into one given by the first spherical harmonic. Now it is clear what should happen in the dual field theory. The $\mathrm{SYM}_4$ on $\mathbf{S}^3$ must undergo a phase transition to a phase where the SO(6) global symmetry is broken to the subgroup SO(5). Thus, from the field theory viewpoint, the Gregory-Laflamme instability is simply a symmetry breaking phase transition.

This was argued previously by Gubser and Mitra [1]; however, these authors considered only the black brane cases, as discussed above. In fact, the main reason of [1] for considering noncompact symmetries (apart from excluding the cases which are dynamically stable despite being thermodynamically unstable) rests on the fact that by the AdS/CFT duality, the Gregory-Laflamme instability of a black hole horizon may be viewed as a phase transition in the dual gauge theory; but real phase transitions can only occur (or are well-understood) in noncompact systems, where there are no finite volume effects. Yet it is by now well-established that the Gregory-Laflamme instability persists even in compact systems, and moreover in gravitational systems having a dual field theoretic description. Thus, one can ask, what kind of a transition do these instabilities correspond to?

A related question in this context is, given that there are other stable inhomogeneous configurations that are likely candidates for the end-point, can one make a statement about what order phase transition in the thermal ensemble is one encountering at the onset of the instability. One would have thought that the phase transition would have to be a second order transition since the existence of the threshold unstable mode would imply that one can smoothly deform the solution away from the initial configuration[11]. However, whereas one could make the same argument in the context of the Gregory-Laflamme black string, numerical investigations in this direction [20] seem to indicate that for the black string, the transition is first order. It would be interesting to understand this issue better, particularly for the Schw-$AdS_5 \times \mathbf{S}^5$ case under consideration.

Now what about the higher harmonics $\ell > 1$? If we consider $r_+ < 0.426L$, then, according to our conjecture, we expect a dynamical instability with the linearized mode (which is still a dipole on the 5-sphere) growing exponentially in time[12]. In other words, if we already allow for the $\ell = 1$ mode to be the progenitor of the phase transition, then we no longer have the right to condense the other modes. The issue of stability of the condensed phase should then be discussed from the deformed solution point of view. However, if we introduce some boundary conditions at infinity to disallow the $\ell = 1$ mode from condensing, then we could condense the higher harmonics to break the symmetry. This means that the gauge theory should admit an infinite tower of phase transitions (with appropriately restricted symmetries), for smaller and smaller values of $r_+$, the first few of which are given in table 1. It is perhaps surprising that such a (relatively) simple gauge theory exhibits such a rich behaviour; but this is just another remarkable consequence of the AdS/CFT duality.

One could have also considered other internal spacetimes, such as $\mathbb{RP}^5$ or $\mathbb{T}^{1,1}$, and modulo details depending on the eigenfunctions of the internal spacetime, which would affect where exactly the instability starts out, we would still have symmetry broken phases for the dual gauge theory.

## 5. Cosmological horizons

So far, we have considered only spacetimes with black hole event horizons, as defined by the boundary of the causal past of $\mathcal{I}^+$; for static spacetimes, this is identical to the marginally trapped surface, i.e., the apparent horizon. Either definition of the horizon

[11]Note that second order phase transition is not a-priori disallowed as it would be if the small black hole in fact localized on the 5-sphere, as was commonly believed until the work of [7]. Since these authors showed that the horizon can't fragment, but rather stabilizes in some nonuniform state, this nonuniform state could possibly be smoothly matched to the uniform state.

[12]Although we have not done so, it would be an interesting and useful exercise to check this numerically.

is observer-independent. But there also are well-known examples of causally nontrivial spacetimes with observer-*dependent* horizons. The simplest and most important of these is the de Sitter (dS) spacetime, which has a cosmological horizon. This is a maximally symmetric solution to Einstein's equation with a positive cosmological constant $\Lambda$. In static coordinates, the metric of 4-dimensional de Sitter spacetime is

$$ds^2 = -\left(1 - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2\, d\Omega_2^2\,. \tag{5.1}$$

For any geodesic observer (which in the above coordinates would sit at $r = 0$), the cosmological horizon is centered on the observer, at radius $r_c = \sqrt{3/\Lambda}$.

It is by now well-known that in many respects, cosmological horizons behave similarly to the black hole horizons; for example, they are subject to the same laws of black hole mechanics such as the area theorem. One might then expect that they could also be dynamically unstable, in a similar fashion as black hole horizons are. The main question then is, *can observer-dependent horizons be Gregory-Laflamme unstable?*

Naively, one might expect that the answer should be "no", because of the observer dependence: In all the black hole examples, one usually associates the Gregory-Laflamme instability with dynamical deformation of the horizon, i.e., the metric perturbation is largest in a particular region of spacetime. Similarly, one usually considers the perturbations which are spherically symmetric in the black hole directions, around the singular origin. In de Sitter, the origin, and consequently the position of the horizon, is not physically distinguished from any other point; it is rather like a gauge choice. Thus, one can ask, how can a physical instability depend on a "gauge choice", namely where the observer sits? The answer is that it of course can't. In particular, the candidate metric perturbations leading to a Gregory-Laflamme instability must preserve the de Sitter symmetry group. This can only be done by changing the size of de Sitter. One in fact can think of this somewhat analogously to considering the spherically symmetric perturbations of the dimensionally reduced black hole, while the size of the horizon becomes nonuniform in the transverse directions. This intuition provides a hint as to what type of perturbations we should be looking for.

In analogy with our previous discussion of black holes, we will focus on spacetimes wherein the cosmological horizon is initially uniformly smeared over some internal space. For simplicity, we will work with Einstein gravity with positive cosmological constant.[13] This requires that the "transverse" space is positively curved; in particular, we can no longer consider $\mathbf{S}^1$ as in the case of the Gregory-Laflamme black string. A typical candidate would then be, for example, $dS_4 \times \mathbf{S}^2$.

For generality, we will work in a $d$ dimensional spacetime, dividing $d$ up into a de Sitter part of dimension $p$ and an internal part $Y$ of dimension $d-p$. The constraint will be that $dS_p \times Y^{d-p}$ be a solution to Einstein's equations in $d$ dimensions with fixed cosmological constant $\Lambda$ i.e., solutions to $R_{MN} - \frac{1}{2} R g_{MN} + \Lambda g_{MN} = 0$. Taking the trace, we obtain

[13] This ignores the difficulties string theory has in constructing and dealing with such spacetimes, however the examples we'll study here will still be of interest from the purely general relativistic point of view.

$R = \frac{2d}{d-2}\Lambda$, which implies

$$R_{MN} = \frac{2}{d-2}\Lambda\, g_{MN}\,. \qquad (5.2)$$

Two interesting $Y$s for us will be $\mathbf{S}^q$ and $\mathbf{S}^q \times \mathbf{S}^r$, with $q = d-p$ and $q+r = d-p$, respectively.

Let us first consider the metric for $dS_p \times \mathbf{S}^q$ for general $p$ and $q$. We can write this as

$$ds^2 = -\left(1 - \frac{r^2}{L_p^2}\right) dt^2 + \left(1 - \frac{r^2}{L_p^2}\right)^{-1} dr^2 + r^2\, d\Omega_{p-2}^2 + \tilde{L}_q^2\, d\tilde{\Omega}_q^2 \qquad (5.3)$$

where $L_p$ is the size of the de Sitter part and $\tilde{L}_q$ of the sphere part; the two being related through Einstein's equation in terms of $p$ and $q$ as follows: $L_p^2 = (p-1)\,(d-2)/2\Lambda$, and $\tilde{L}_q^2 = (q-1)\,(d-2)/2\Lambda$.

Let us now try to see if our idea of euclidean negative modes can be applied here. According to our conjecture, if the "black hole" ($\mathbf{B}_m$) part of the spacetime has a negative euclidean eigenvalue $\lambda_E$, which matches precisely with the eigenvalue $-\mu^2$ of the Laplacian on the "transverse" space $X^n$, then there exists a threshold unstable mode. In the present case, we are taking $\mathbf{B}_m \equiv dS_p$ and $X^n \equiv \mathbf{S}^q$. The euclidean continuation of $p$-dimensional de Sitter is a $p$-sphere; but the euclidean Lichnerowicz operator evaluated in the background of $\mathbf{S}^\mathbf{p}$ has no negative eigenvalue. This immediately implies that $dS_p \times \mathbf{S}^q$ is *stable*[14] to the Gregory-Laflamme clumping on the $\mathbf{S}^q$. Note that we cannot really appeal to any kind of thermodynamic argument in this case, since although de Sitter spacetime has a well defined entropy, the notion of specific heat is murky since it is hard to define the energy of de Sitter space.

Before proceeding to the second example, let us briefly compare this conclusion with what one might naively expect from global entropy considerations. The initial intuition might lead one to expect the result we found above: Einstein's equation fixes the sizes of de Sitter and the sphere to be comparable, which in the case of the compact black string would suggest stability. A more detailed calculation, on the other hand, shows that $dS_p \times \mathbf{S}^q$ is *not* the most entropically favorable configuration if $q \neq 0$. In particular, the horizon area for $dS_p \times \mathbf{S}^q$ is given by[15]

$$A_{p+q} = 2\pi^{d/2} \left(\frac{d-2}{2\Lambda}\right)^{\frac{d-2}{2}} \frac{(p-1)^{\frac{p}{2}}}{\Gamma\left(\frac{p+1}{2}\right)} \frac{(q-1)^{\frac{q}{2}}}{\Gamma\left(\frac{q+1}{2}\right)}\,. \qquad (5.4)$$

One can easily see that the area is maximized for pure de Sitter. A global entropy argument would then suggest that $dS_p \times \mathbf{S}^q$ should localize[16] into pure $dS_{p+q}$. As we have seen before, and confirmed above, this argument is misleading.

[14]While there are claims [21, 22], that in certain cases $dS_p \times \mathbf{S}^q$ is unstable, such instabilities would not correspond to symmetry breaking along the sphere, and therefore do not fall under the category of Gregory-Laflamme instabilities we are considering.

[15]This uses the formula for the total horizon area $A_{p+q} = A_p \times A_q = \omega_{p-2}\, L_p^{p-2} \times \omega_q\, \tilde{L}_q^q$ where the area of a unit $S^{n-1}$ is $\omega_{n-1} \equiv \frac{2\pi^{n/2}}{\Gamma(n/2)}$. In the special case where $q = 0$, one has to divide the area by 2, because if we have no internal space, we want to take the area $\omega_0 = 1$ rather than $\omega_0 = 2$.

[16]Of course, one can apply the arguments of Horowitz and Maeda [7] to show that, just as for the black string, the horizon cannot really fragment.

Our failure to find the cosmological analog of a threshold unstable mode is revealing, in that we can now try to engineer a configuration that is more likely to exhibit an instability. The important point to notice is that while $\mathbf{S^p}$ does not have negative eigenmode, $\mathbf{S^p} \times \mathbf{S}^q$ *does*. One should perhaps reiterate that while this means that the euclidean version of $dS_p \times \mathbf{S}^q$ has a negative mode, this does not mean that $dS_p \times \mathbf{S}^q$ is dynamically unstable, just as the negative euclidean mode for Schwarzschild black hole does not imply dynamical instability of the same. Instead, this motivates us to consider $dS_p \times \mathbf{S}^q \times \mathbf{S}^r$, with the $d(\equiv p+q+r)$-dimensional metric

$$ds^2 = -\left(1 - \frac{r^2}{L_p^2}\right) dt^2 + \left(1 - \frac{r^2}{L_p^2}\right)^{-1} dr^2 + r^2\, d\Omega_{p-2}^2 + \tilde{L}_q^2\, d\tilde{\Omega}_q^2 + \bar{L}_r^2\, d\bar{\Omega}_r^2 \tag{5.5}$$

where the sizes of the three components are again given by $L_p^2 = (p-1)(d-2)/2\Lambda$, $\tilde{L}_q^2 = (q-1)(d-2)/2\Lambda$, and $\bar{L}_r^2 = (r-1)(d-2)/2\Lambda$.

We shall now argue that this spacetime is unstable. In terms of the notation of section 3, we wish to take $B_m \equiv dS_p \times \mathbf{S}^q$, $X^n \equiv \mathbf{S}^r$; and the trick is to use the fact that the euclidean space $\mathbf{S^p} \times \mathbf{S^q}$ has a negative mode. Physically, this mode corresponds to increasing the radius of one of the spheres and simultaneously decreasing the radius of the other. Let us therefore postulate the following ansatz for small fluctuations for the lorentzian spacetime:

$$\begin{aligned} h_{\mu\nu} &= \frac{1}{p} g_{\mu\nu} \phi(\bar{\Omega}_r) \\ h_{ab} &= -\frac{1}{q} g_{ab} \phi(\bar{\Omega}_r) h_{\alpha\beta} = 0 \end{aligned} \tag{5.6}$$

where we use $(\mu,\nu)$, $(a,b)$ and $(\alpha,\beta)$ to denote indices along $dS_p$, $\mathbf{S^q}$ and $\mathbf{S^r}$, respectively, and later $(M,N)$ to denote indices of the full $d$ dimensional spacetime. The function $\phi(\bar{\Omega}_r)$ is a function only of the coordinates on the $r$-sphere, because we are only changing the relative radii, but keeping the de Sitter and $q$-spherical symmetries.

The fluctuations are easily seen to be traceless and satisfy the transverse gauge condition $\nabla^M h_{MN} = 0$. The equation of motion is the linearization of (5.2) i.e., $R^{(1)}_{MN}(h) = \frac{2}{d-2}\Lambda h_{MN}$ and the linearized Ricci tensor is the same as in (4.3). With our gauge choice, this simplifies to

$$R^{(1)}_{MN} = -\frac{1}{2}\left[\left(\nabla_\mu\nabla^\mu + \nabla_a\nabla^a + \nabla_\alpha\nabla^\alpha\right) h_{MN} - 2R_{MPQN}h^{PQ} - R^P{}_M h_{NP} - R^P{}_N h_{MP}\right]. \tag{5.7}$$

Using our ansatz (5.6) the first two terms vanish, and furthermore it is easy to check that the maximally symmetric nature of each of the spaces in the product spacetime causes the terms proportional to the Riemann tensor and the Ricci tensor to cancel, so that the equation of motion reduces simply to

$$-\frac{1}{2}\nabla_\alpha\nabla^\alpha \phi(\bar{\Omega}_r) = \frac{2}{d-2}\Lambda\phi(\bar{\Omega}_r)\,. \tag{5.8}$$

One can choose $\phi(\bar{\Omega}_r)$ to be $\ell$th spherical harmonic, with eigenvalue $-\ell(\ell+r-1)/\bar{L}_r^2$, and using the expression for $\bar{L}_r^2$, we obtain

$$\ell(\ell+r-1) = 2(r-1)\,. \tag{5.9}$$

The only allowed solutions to (5.9) are ones where both $\ell$ and $r$ are integers. A simple solution is $(r, \ell) = (2, 1)$; in fact, it is also easy to check that this is the *only* solution for finite $r$. The establishes for us the threshold unstable mode.

The upshot is that $dS_p \times \mathbf{S}^q \times \mathbf{S}^2$ is *unstable* to small fluctuations, which change the radius of the $dS_p$ and $\mathbf{S}^{\mathbf{q}}$ commensurately, the functional form of the change being a dipole on the $\mathbf{S}^2$. Moreover, this is true for all dimensions $p$ and $q$ so long as $p + q = d - 2$ and $\Lambda$ are fixed. It is interesting to note that despite the fact that the internal space in positively curved, which as [23] argued, would tend to stabilize the spacetime, we nevertheless find an instability.

Although we have exhibited marginal (static) instability of $dS_p \times \mathbf{S}^q \times \mathbf{S}^2$; as argued above, we cannot obtain a genuine dynamical instability while maintaining the full de Sitter invariance. However, if we consider perturbations which do break this invariance, such argument no longer holds. Moreover, we can start with cosmological spacetimes which do not respect the full de Sitter group, such as the Schwarzschild-de Sitter spacetime. This spacetime now has two horizons, the black hole and the cosmological one; and either one could be subject to the Gregory-Laflamme instability. In fact, one might expect that if the black hole is dynamically unstable and if the black hole horizon is sufficiently close to the cosmological horizon, then the cosmological horizon must also exhibit dynamical instability, because otherwise the growth of the black hole horizon area without the accompanying growth of the de Sitter horizon area would threaten to violate the holographic bound [24]. While it is therefore plausible that a cosmological horizon can exhibit a dynamical instability, we leave this for future investigation.

## 6. Discussion

One fundamental principle governing the dynamical evolution of any system is the second law of thermodynamics, i.e., the total entropy must be nondecreasing with time. For isolated black holes, this is identified with the area theorem, stating that the area of the event horizon cannot decrease in any physically allowed process. While the formulation is quite simple, its consequences are far-reaching. One might therefore be inclined to expect that an entropy (or horizon area) based argument could determine the stability of a given solution.

As we saw, there is a hierarchy of entropy-based arguments: First, the *global* argument, such as what [6] used to motivate the instability of a black string, rests on the existence of a state with higher entropy. Namely, if there exists a configuration with entropy higher than that of the initial configuration, but with all the same conserved quantities, then the initial state must be unstable. However, this criterion is not satisfactory in considering a classical process, as such a state may be reached from the initial one only via quantum tunneling.

This observation led [1] to consider the *local* version, namely a state would be unstable if the system gains entropy by locally deforming (usually by "clumping") the horizon, while preserving all the conserved quantities; or stated more mathematically, if the Hessian of the

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entropy with respect to the extensive variables would have positive eigenvalues. However, as we argued, even this does not specify the complete picture. Only for solutions with noncompact translational invariance does the conjecture seem to hold!

In this paper, we have gone one step further, and proposed a criterion for all classes of static spacetimes which have a direct product structure. This conjecture has some features of both the local and the global arguments. While the criterion relates local thermodynamic instability to a dynamical Gregory-Laflamme type instability, as in the Gubser-Mitra conjecture, it is fully geometric and incorporates the information about the whole spacetime. We have checked a typical and useful example, the Schwarzschild-$AdS_5 \times \mathbf{S}^5$ black hole, and found explicitly where the onset of instability occurs. We have also used our conjecture to motivate an instability of a cosmological horizon, in the spacetime of the form $dS_p \times \mathbf{S}^q \times \mathbf{S}^2$.

However, while we have attempted to address a wider class of solutions, we still do not understand the most general types of horizons. Most importantly, we have not addressed the cases where the horizon does not have a product structure. While such cases would be interesting and useful to get a handle on, there are only very few solutions which are known explicitly. Among these are the $d$-dimensional Schwarzschild and its rotating and charged generalizations, but these are classically stable. There are also more complicated explicit (stationary) solutions, such as those considered by [25], but they tend to be too complicated to see their stability. Others, such as the solutions which correspond to the endpoints that most of the instabilities evolve to [7], or those considered later by [26], are not known explicitly, though we may understand some of their properties.

We have been discussing stability of spacetimes with horizons. Part of the motivation rested on the fact that the special properties exhibited by horizons allow one to hope for a simple formulation of a universal criterion for instability. While we have not been able to address the most general static spacetimes with horizons, we proposed a generalization of the previous understanding to all black hole product spacetimes. We can now look back and ask, to what extent was the presence of a horizon really necessary for our proposal? More specifically, one might conjecture that *any* product spacetime of the form $\mathbf{B}_m \times X^n$, where the $\mathbf{B}_m$ part of the spacetime has a negative euclidean eigenvalue which matches precisely with the eigenvalue of the Laplacian on the "transverse" space $X^n$, has a threshold unstable mode. While a-priori there seems nothing wrong with this proposal, it is difficult to find (static) spacetimes with negative euclidean modes in the absence of horizons.

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